ON SETS HAVING FINITELY MANY POINTS OF LOCAL NONCONVEXITY AND PROPERTY P_m

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ABSTRACT

If a point q of S has the property that each neighborhood of q contains points x and y such that the segment xy is not contained by S, q is called a point of local nonconvexity of S. Let Q denote the set of points of local nonconvexity of S. Tietze's well known theorem that a closed connected set S in a linear topological space is convex if $Q=\phi$ is generalized in the result: *If S is a closed set in a linear topological space such that $S \sim Q$ is connected and $|Q| = n < \infty$, then S is the union of $n+1$ or fewer closed convex sets.* Let k be the minimal number of convex sets needed in a convex covering of S. Bounds for k in terms of m and n are obtained for sets having property P_m and $|Q| = n$.

A set S having at least $m \geqq 2$ points is said to be *m-convex*, with the property of m-convexity referred to as *property P_m*, if among each m points of S there exists at least one pair such that the segment joining that pair lies in S. Valentine's theorem [5] that a closed 3-convex set in E^2 is the union of three or fewer convex sets (a result obtained for a special class of closed 3-convex sets in E^n by E. Buchman [1]), has sparked research in several directions. The general concept of m-convexity is discussed fully in the authors' paper [2], where it is shown that Valentine's result for $m = 3$ does not generalize in any simple way to higher values of m and the uncertainty of a bound for the number of convex subsets needed to cover a closed planar m-convex set is discussed.

Research on the problem ultimately involves the concept of "local convexity" in sets. We define a weak type of local convexity, one which is meaningful in any linear topological space (see [6], pp. 48–49, Definition 4.2): A point x in a set S is called a *point of local convexity of* S (alternatively, *S is locally convex at x*) if there exists some neighborhood U of x such that if $y \in S \cap U$ and $z \in S \cap U$ then

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the segment yz lies in S. S is *locally convex* iff it is locally convex at each point, and if S fails to be locally convex at some point q in S, that point is called a point of *local nonconvexity* (lnc) of S.

The classical theorem involving this concept was first proved for sets in E^n by Tietze, later extended to the following form by other authors (see [6], p. 49 Theorem 4.4; a new proof involving the concept of m-convexity appears in [2]):

TIETZE'S THEOREM. In any linear topological space, a closed, connected locally convex set is convex.

The chief motive for the work done in this paper was to obtain an upper bound on the minimum number of convex sets required in a convex covering of S. The authors were early able to establish that if a closed set is m-convex and has one lnc point, then it is the union of $m - 1$ or fewer convex sets. Slightly more effort produces the bound $(m - 1)\,[(n + 1)/2]$ for closed m-convex sets having n points of local nonconvexity. In seeking results for sets having more than one lnc point the following generalization of Tietze's theorem was found, interesting by itself:

THEOREM. *If S is a closed set in a linear topological space and Q is the set of* lnc *points of S, with $|Q| = n < \infty$ and $S \sim Q$ connected, then S is either convex or planar, and in either case S is the union of $n+1$ or fewer convex sets.*

We shall assume throughout that the space is Hausdorff, and all results are valid for linear topological spaces unless otherwise stated. We shall use essentially the terminology and notation established in [6]. For convenience we denote the segment, open segment, and half open segments joining x and y by xy, $(xy) \equiv xy \sim \{x\} \sim \{y\}$, $(xy] \equiv xy \sim \{x\}$, and $[xy) \equiv xy \sim \{y\}$, respectively. The line determined by x and y (the set $\{\lambda x + \mu y : \lambda + \mu = 1\}$) will be denoted $L(x, y)$, the ray from x to y (the set $\{\lambda x + \mu y : \lambda + \mu = 1,\ \mu \geqq 0\}$) will be designated $R(x, y)$, and the angle $A(x, y, z)$ is defined as the set $R(y, x) \cup R(y, z)$. The *x-star* of S, also called the *local kernel of S at x*, is denoted S_x and is defined to be the set consisting of all points y in S which can see x via S (that is, $xy \subset S$). Thus, S is starshaped with respect to x iff $S = S_x$. We denote the convex hull of a set S by conv S, with the topological interior, closure, and boundary of S being written as int S, cl S, and bd S, respectively. Finally, we denote the set of lnc points of S (as defined above) by Q. We shall assume throughout that Q is of finite cardinality, and that S is closed and connected.

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1. Sets having finitely many points of local nonconvexity

Tietze's theorem provides the main tool for proving the results needed to generalize it. Some of our results are to be found in Valentine [7] and have been included here for the sake of completeness. We begin with the following key result:

LEMMA 1. *If $xy \cup yz \subset S$ and no point of Q lies in conv $\{x,y,z\} \sim xz$, then conv $\{x,y,z\} \subset S$.*

Proof. Choose any point $u \in (xy)$ and $v \in (yz)$. Since $\text{conv}\{u,v,y\} \subset \text{conv}\{x,y,z\} \sim xz$, $\text{conv}\{u,v,y\} \cap Q = \emptyset$. Let T be the component of $S \cap \text{conv}\{u,v,y\}$ containing $uy \cup yv$. Since an lnc point of T would clearly be an lnc point of S, T is closed, connected, and locally convex. By Tietze's theorem T is convex. Hence $uv \subset T \subset S$ for all $u \in (xy)$ and $v \in (yz)$. Since S is closed, this implies $\text{conv}\{x,y,z\} \subset S$, as desired.

If a connected set is locally convex, standard arguments show it is polygonally connected. Thus it is clear that any component of $S \sim Q$ is polygonally connected. This observation is used in proving the next lemma, a result which appears also in [7]; note that our proof applies to any closed connected set S with a nonempty set of lnc points.

LEMMA 2. $S = \cup_{q \in Q} S_q$.

PROOF. Let $x \in S$, $x \notin Q$, and let W be the component of $S \sim Q$ containing x. Since $Q \cap \text{cl } W \neq \emptyset$, the result follows immediately if W (and thus cl W) is starshaped with respect to x, so assume there is some point $x' \in W$ which x cannot see via W. By the polygonal connectedness of W there exist points y and z in W such that $xy \cup yz \subset W$ but $xz \not\subset W$. It follows by Lemma 1 that $Q' \equiv Q \cap \text{conv}\{x,y,z\} \neq \emptyset$. Consider $\cap_u \text{conv}\{x,y,u\}$, where the intersection is taken over all $u \in yz$ such that $xu \cap Q' \neq \emptyset$. Since S is closed, Q' is compact and hence there exists $u_0 \in yz$ such that $\cap_u \text{conv}\{x,y,u\} = \text{conv}\{x,y,u_0\}$, and there exist $q_0 \in Q' \cap xu_0$. Since $\text{conv}\{x,y,u_0\} \subset S$, $xq_0 \subset S$ and $x \in S_{q_0}$.

We say that S is *locally starshaped* iff for each point $x \in S$ there exists a relative neighborhood of x in S which is starshaped with respect to x. (Thus, any open set in a topological linear space is locally starshaped.) The fact that a set S having the properties assumed for this section is locally starshaped may be obtained in the following manner: Since there is no problem for a point x of local convexity, let $x = q_1 \in Q$ and suppose there is a net $\{x_i\}_{i \in D}$ converging to q_1 such that $x_i q_1 \not\subset S$ for all $i \in D$. Since $x_i \in \cup_{q \in Q} S_q$ and Q is finite, $x_i q_2$ is frequently in S for some $q_2 \in Q$, $q_2 \neq q_1$. Let U be a neighborhood of q_1 devoid of points of $Q \sim \{q_1\}$. Then some subnet $\{x_j\}_{j \in E}$ is such that $x_j \in U$ and $x_j q_2 \subset S$ for all $j \in E$. Choose a point $y \in (q_1 q_2)$ such that $q_1 y \subset U$, let $z = \lambda q_1 + \mu q_2$ be any point on $(q_1 y)$, with $\lambda + \mu = 1$ and $0 < \lambda < 1$, and define $z_j = \lambda x_j + \mu q_2$. Since $\lim_{j \in E} z_j = z \notin Q$ and $S \sim Q$ is locally starshaped $z_j z$ is eventually a subset of S and $z_j \in U$. By Lemma 1 there is a point $q_j \in \operatorname{conv}\{x_j, z_j, z\}$ or $q_j \in \operatorname{conv}\{x_j, q_1, z\}$, $q_j \neq q_1$, and since for certain non-negative α_j, β_j, γ_j with $\alpha_j + \beta_j + \gamma_j = 1$, $q_j = \alpha_j x_j + \beta_j z_j + \gamma_j z$, or $q_j = \alpha_j x_j + \beta_j q_1 + \gamma_j z$, it may be assumed without loss of generality that for some non-negative η, ν with $\eta + \nu = 1$, $\lim_{j \in E} q_j = \eta q_1 + \nu z = q_3$ for some $q_3 \in Q$. But then $q_3 \in q_1 z \subset U$, and by definition of U, $q_3 = q_1$. But since $\{q_j\}$ converges to q_3, q_j is eventually in U, a contradiction. (Simple examples in E^2 show that S need not be locally starshaped if Q is not finite.) Since connected, locally starshaped sets are obviously polygonally connected, we have proved

LEMMA 3. *S is locally starshaped and polygonally connected. Moreover, each component of $S \sim Q$ is locally convex, and its closure is locally starshaped and polygonally connected.*

We shall now consider the case when $S \sim Q$ is connected. It is interesting that S is convex in certain cases by virtue of its dimension.

LEMMA 4. *If $S \sim Q$ is connected and S does not lie in a plane, then S is convex.*

PROOF. Consider any two points x and y in $S \sim Q$ and let $x = x_0, \cdots, x_k = y$ be the vertices of a polygonal arc P in $S \sim Q$ joining x and y having the minimal number of vertices, and assume $k \geqq 2$. Then no three consecutive vertices of P are collinear, and we may define i as the largest integer such that $x_0, \cdots, x_i$ lie in the plane π of x_0, x_1, x_2, $2 \leqq i \leqq k$. Suppose $i < k$; then no three of the points x_{i-2}, x_{i-1}, x_i, and x_{i+1} are collinear, and since $x_{i-1} x_i \cap Q = \emptyset$, it follows that $\operatorname{conv}\{x_{i-1}, x_i, u\} \cap Q = \emptyset$ for some $u \in x_i x_{i+1}$. By Lemma 1, $\operatorname{conv}\{x_{i-1}, x_i, u\} \subset S \sim Q$. For all but finitely many $v \in x_i u$, $\operatorname{conv}\{x_{i-2}, x_{i-1}, v\} \cap Q = \emptyset$ which implies $\operatorname{conv}\{x_{i-2}, x_{i-1}, v\} \subset S \sim Q$. Therefore, the points $x_0, \cdots, x_{i-2}$, v, $x_{i+1}, \cdots, x_k$ determine a polygonal arc in $S \sim Q$ joining x and y having only $k-1$ vertices, contradicting the minimal property of P. Hence $i = k$ and P lies in π.

Since $S \sim Q$ does not lie in π there exists a point $z \in S \sim Q \sim \pi$, and by considering a polygonal arc in $S \sim Q$ with vertices $y = x_k$, $x_{k+1}, \cdots, x_l = z$, there is a maximal $j \geqq k$ such that $x_j \in \pi$. Let F be the 3-dimensional flat containing $P \cup x_k x_{k+1} \cup \cdots \cup x_j x_{j+1} \equiv P'$ and let S' be the component of $S \cap F$ containing P'. It is clear that S' is closed, and if Q' is the set of lnc points of S', $Q' \subset Q$; moreover, $P' \subset S' \sim Q'$. Thus some plane π' exists which is parallel to π, cuts $x_j x_{j+1}$ at some point z', and is sufficiently close to π that no member of Q' lies strictly between π and π'. It may be proved by induction that $z'x_j$, $z'x_{j-1}, \cdots$, $z'x_0$ all lie in S' (applying Lemma 1). Again by Lemma 1, $xz' \cup z'y \subset S'$ implies $xy \subset S' \subset S$, a contradiction. Hence, $k = 1$ and S is convex.

The next lemma yields a topological property of S which will be useful later.

LEMMA 5. *If S is finite-dimensional and $S \sim Q$ is connected, then $S =$ cl(int S), the interior being taken relative to the minimal flat containing $S \sim Q$.*

PROOF. The statement is obvious if S, and therefore $S \sim Q$, is convex. Otherwise, Q is nonempty and by Lemma 4, S is planar. Since S is closed we have $\mathrm{cl}(\mathrm{int}\, S) \subset \mathrm{cl}\, S = S$. To reverse the inclusion we have only to show that $S \sim Q \subset \mathrm{cl}(\mathrm{int}\, S)$, since $S = \mathrm{cl}(S \sim Q)$. Suppose $x \in S \sim Q$, and let U be a neighbourhood of x such that $S \cap U$ is convex. If $\dim(S \cap U) = \dim S$, then $x \in \mathrm{cl}(\mathrm{int}\, S)$. If $\dim(S \cap U) < \dim S$, then necessarily $\dim(S \cap U) = 1$ and $\dim S = 2$ since S is connected. Since $S \cap U$ is convex $S \cap U$ is free of points of Q, and since $S \cap U$ is a segment, ray, or line we may take the maximal convex subset T of S containing $S \cap U$ and T will be a segment, ray, or line. But then T would be in all cases a nontrivial component of $S \sim Q$, contradicting the assumption that $S \sim Q$ is connected.

We now consider the situation when Q contains a single element q and $S \sim Q = S \sim \{q\}$ is connected. Then Lemma 4 implies that S is a subset of E^2. A sequence of observations will further show S to be, in this case, the union of exactly *two* convex sets. Choose $x \in S$, $y \in S$ such that $xy \not\subset S$; by Lemma 2, $xq \cap qy \subset S$, and we define T to be the closure of the geometric interior of the angle $A(x, q, y)$ (see Fig. 1).

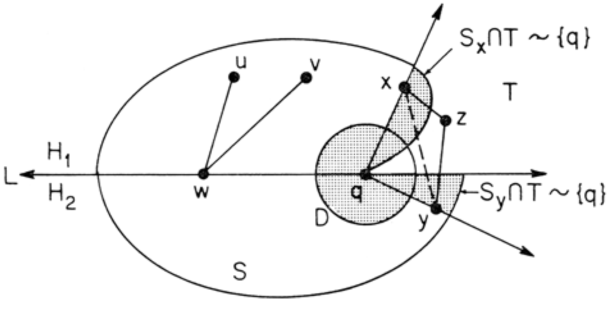


Fig 1.

(1) No point $z \in S \cap T \sim \{q\}$ can see both x and y via S, for Lemma 1 would imply that $xy \subset S$.

(2) $S_x \cap T \sim \{q\}$ and $S_y \cap T \sim \{q\}$ are convex sets (by the same reasoning as in (1)), and $S \cap T = (S_x \cap T) \cup (S_y \cap T)$.

(3) By (1) $S_x \cap T \sim \{q\}$ and $S_y \cap T \sim \{q\}$ are disjoint; accordingly, there exists a separating line L through q such that if H_1 and H_2 are the open half-planes determined by L, with $x \in H_1$ and $y \in \mathrm{cl}\ H_2$ or $x \in \mathrm{cl}\ H_1$ and $y \in H_2$, then $S_x \cap T \subset \mathrm{cl}\ H_1$ and $S_y \cap T \subset \mathrm{cl}\ H_2$.

(4) Since q is not a cut-point of S there exists a $w \in S \cap L \sim T$ (for, if $S \cup L \sim T = \emptyset$ then $(S \cap H_1) \cup (S_x \cap T \sim \{q\})$ would be a closed and open proper subset of $S \sim q$). It follows that $S = S_w$ since Lemma 1 implies that any polygonal arc in $S \sim \{q\}$ joining w with a point $u \in S \sim \{q\}$ may be reduced to uw.

(5) If u and v belong to $S \cap H_1$ then $uw \cup wv \subset S$ and Lemma 1 implies $uv \subset S \cap H_1$; similarly for $S \cap H_2$.

Thus, we have proved:

LEMMA 6. *If $S \sim Q$ is connected and Q consists of exactly one point, S is planar and is the union of the two convex sets* $\mathrm{cl}(S \cap H_1)$ *and* $cl(S \cap H_2)$, *as defined above.*

It will be convenient to introduce the following terminology for sets in E^2: point $q \in S$ is called an *essential* point of local nonconvexity of S iff for every neighborhood U of q there is at least one component W of $S \cap U \sim \{q\}$ such that q is an lnc point of cl W; an lnc point that is not essential is called an *inessential* point of local nonconvexity. Let us now consider the local behavior of S at an essential lnc point q. Let D be a closed circular disk centered at q which is sufficiently small to exclude all other lnc points of S, and such that $S \cap D$ is starshaped with respect to q. The closures of the components of

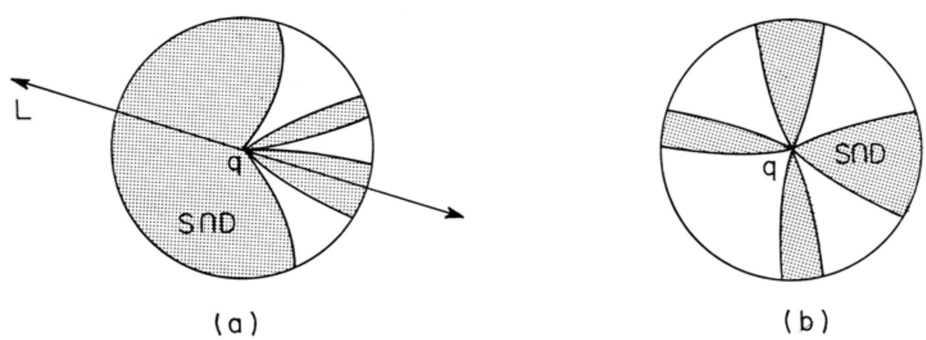


Fig. 2.

$S \cap D \sim \{q\}$ can have at most one lnc point, and in that event, it must be q; moreover, at most one of these components can have q as lnc point. Hence, since q is essential, precisely one component W of $S \cap D \sim \{q\}$ has q as an lnc point of its closure, and the closures of all other components (if they exist) are convex by Tietze's theorem (see Fig. 2-a). By the observations which led to Lemma 6, cl W is the union of two convex sets defined by a (weak) separating line L. We call line L a *separating line* at q. (Fig. 2-b illustrates the case when q is inessential.)

The following argument will show that if $S \sim Q$ is connected and $|Q| > 0$, then S always has at least one essential lnc point: Let $q_1, \cdots, q_n$ be the lnc points of S, assumed to be inessential for sake of argument. Since S is planar, let $D = D_k$ be a closed circular disk of radius $r_k > 0$ centered at an lnc point $q = q_k$, $1 \leqq k \leqq n$ (without loss of generality we may assume $r_1 = \cdots = r_k = r$), and put $C = bd\ D$. For sufficiently small r, $S \sim D$ is connected, and the closures of the components W_α $(\alpha \in A)$ of $S \cap D \sim \{q\}$ are convex (since q is inessential). Since $S \sim \{q\}$ is connected each W_α meets C. Let $K_\alpha \subset C$ be a (closed) circular arc on C of minimal length that contains $W_\alpha \cap C$, with endpoints x_α and y_α. The arc K_α must be of length $\Lambda(K_\alpha) \leqq \pi r$ or else $q \in \text{int } S$. Define $B = \{\alpha : \Lambda(K_\alpha) < \pi r\}$, and for $\alpha \in B$ let W'_α denote the component of $W_\alpha \sim x_\alpha y_\alpha$ whose closure contains q. Then if

$$T = \bigcup_{\alpha \in B} \text{cl } W'_\alpha,$$

the set $S \sim T$ will not have q as an lnc point. Doing this for all sufficiently small r and at each lnc point q_k yields sets $T_k(r)$ such that

$$\text{cl}\left(S \sim \bigcup_{k=1}^{n} T_k(r)\right) \equiv S(r), \quad 0 < r < a,$$

has no lnc points and is connected. Thus $\{S(r): 0 < r < a\}$ is a directed family of closed, connected, locally convex sets, thus convex, and consequently $\text{cl}(\cup_{0<r<a} S(r)) = S$ is convex, a contradiction. This proves

LEMMA 7. *If $S \sim Q$ is connected and $|Q| > 0$, then S is planar and has at least one essential lnc point.*

2. Monotone sequences of sets having finite convex coverings

A result which will expedite later proofs is of a fundamental nature. Suppose we have a monotone increasing sequence of sets $S_1 \subset S_2 \subset \cdots \subset S_i \subset \cdots$ each of which is the union of m convex sets (m fixed). An obvious conjecture is that the union $S = \cup_{i=1}^{\infty} S_i$ is itself the union of m convex sets (a similar statement concerning the intersection of a monotone decreasing sequence of sets may be made). These propositions are valid under varying restrictions and seem not to be so readily proved without them. Both propositions are true if it is required that S and each S_i are topologically closed, and if the space is second countable.

We make use of the Hausdorff limit and a well known theorem in topology which states that any sequence of subsets of a second countable topological space contains a topologically convergent subsequence (see C. Kuratowski [4], p. 246, Theorem VIII). It is an easy matter to show that if the sequence consists of *convex* sets the set to which the sequence converges is convex (one uses here the fact that the definition of the Hausdorff limit demands it to be topologically closed). The second countability assumption requires that the linear space be at least separable and metrizable (the two propositions are therefore valid for separable normed linear spaces). We have, then, the following lemma:

LEMMA 8′. *Each sequence of convex sets in a separable and metrizable linear topological space contains a subsequence which converges topologically to a closed convex set.*

For convenience, we combine the two propositions we mentioned into a single lemma; N will denote the set of positive integers.

LEMMA 8. *Let $\{S_i\}_{i \in N}$ be a nondecreasing sequence of sets in a separable and metrizable linear topological space, each of which is the union of m convex sets. Then the set*

$$S = \mathrm{cl} \bigcup_{i \in N} S_i$$

is also the union of m convex sets. [If the sequence is non-increasing, then the set

$$S = \bigcap_{i \in N} \mathrm{cl}\, S_i$$

is the union of m convex sets.]

Proof. By hypothesis we may suppose that for each $i \in N$

$$S_i = \bigcup_{j=1}^{m} C_{ij} \qquad \left[\mathrm{cl}\, S_i = \bigcup_{j=1}^{m} C_{ij}\right],$$

where C_{ij} is convex, $j = 1, \cdots, m$. Consider the sequence $\{C_{i1}\}_{i \in N}$. By Lemma 8′ there exists a subsequence $\{C_{i1}\}_{i \in N_1}$, $N_1 \subset N$, which converges to a closed convex set C_1. Consider the sequence $\{C_{i2}\}_{i \in N_1}$; it has a subsequence $\{C_{i2}\}_{i \in N_2}$, $N_2 \subset N_1$, converging to a closed convex set C_2. Continuing inductively one can find infinite subsets of N, $N_1 \supset N_2 \supset \cdots \supset N_m$, such that, for each j, the subsequence $\{C_{ij}\}_{i \in N_j}$ converges to a closed convex set C_j. It follows that

$$\lim_{i \in N_m} \{C_{ij}\} = C_j, \qquad j = 1, \cdots, m.$$

Since it is clear that $S = \mathrm{cl}(\cup_{i \in N_m} S_i)$ $[S = \cap_{i \in N_m} \mathrm{cl}\, S_i]$, it is a routine matter to prove that in each case $S = \cup_{j=1}^{m} C_j$.

REMARK. The referee has communicated a proof of Lemma 8 extending it to general sequences of sets in a vector space over an ordered field.

3. A generalization of Tietze's theorem

We now generalize the result of Lemma 6 to obtain the theorem promised earlier.

THEOREM 1. *Let S be a closed set in a topological linear space, with Q the set of points of local nonconvexity of S, and $S \sim Q$ connected. Then if Q has $n < \infty$ members, S may be expressed as the union of $n + 1$ or fewer closed convex sets. Moreover, if S is not convex then S is planar and is the union of $n' + 1$ or fewer closed convex sets, where $n' \leqq n$ is the number of essential lnc points of S.*

PROOF. It is obvious we need only prove the assertion concerning planar sets. The proof will proceed by induction on n'; Lemma 6 implies the case $n' = 1$, for, by the method used in the proof of Lemma 7 the connectedness of $S \sim Q$ can be shown to imply that $n' = n = 1$. Let q be one of the essential lnc points of S. We prove first that no loss of generality results in assuming the existence of more than one separating line at q. Let D be the disk used previously to define the separating line L at q, and let W be the component of $S \cap D \sim \{q\}$ whose closure has q as lnc point. Since cl W is the union of two convex sets, one on each side of L, there exist sequences $\{x_i\}_{i \in N}$ and $\{y_i\}_{i \in N}$ in bd S converging to q such that $(x_i y_i) \cap \mathrm{cl}\, W = \emptyset$ and $T_{i+1} \subset T_i$, where T_i is the geometric interior of the angle $A(x_i, q, y_i)$. Then if $S_i = \mathrm{cl}[S \sim (W \cap T_i)]$, $S_{i+1} \supset S_i$ so that $\{S_i\}$ is a non-decreasing sequence of sets whose union is S. Moreover, each set S_i obviously has q as an essential lnc point and has more than one separating line at q. In view of Lemma 8, it then suffices to prove that each S_i is the union of $n' + 1$ convex sets.

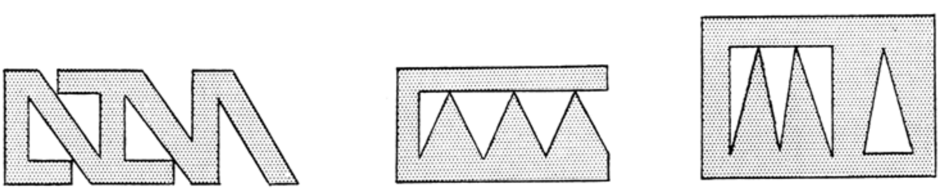
Since Q is finite it is possible to choose a separating line L at q such that $L \cap Q = q$. Take a sequence $\{L_i\}_{i \in N}$ of lines parallel to L, all on the same side of L, such that the width of the closed parallel strip T_i bounded by L_i and L is $1/i$.

Let W be the component of $S \cap D \sim \{q\}$ defined in the preceding paragraph, and denote by W_i the closure of the component of $S \cap \operatorname{int} T_i$ which contains $W \cap \operatorname{int} T_i$. For all sufficiently large i, $T_i \cap Q = \{q\}$; thus it follows that W_i is convex, and $W_{i+1} \subset W_i$ for all i sufficiently large. The set $S_i = \operatorname{cl}(S \sim W_i)$ moreover contains one less essential lnc point than S, and since $S \sim W_i \subset S \sim W_{i+1}$, the sequence $\{S_i\}$ is non-decreasing, with $S = \operatorname{cl}(\cup_{i \in N} S_i)$. The set $Q_i = Q \sim \{q\}$ contains the set of essential lnc points of S_i. For sufficiently large i, S_i has $n' - 1$ essential lnc points. If $S_i \sim Q_i$ is connected, then by the induction hypothesis S_i is the union of $(n' - 1) + 1 = n'$ convex sets. If $S_i \sim Q_i$ is not connected, then it obviously has only two components, say S_{1i} and S_{2i}. The sets cl S_{1i} and cl S_{2i} have n'_{1i}, resp. n'_{2i} essential lnc points, with $n'_{1i} + n'_{2i} = n' - 1$, and in this case S_i is the union of $n'_{1i} + n'_{2i} + 2 = (n' - 1) + 2 = n' + 1$ convex sets. Thus, in either case, S_i may be expressed as the union of $n' + 1$ convex sets and by Lemma 8, S is the union of $n' + 1$ convex sets.

REMARK. If in the above proof $S_i \sim Q_i$ remains connected for all sufficiently large i, then S is the union of n' convex sets. This happens necessarily if q may be chosen on the boundary of some bounded component of the complement of $S \sim Q$. The result may then be extended to the case when there are $k > 0$ bounded components in the complement of $S \sim Q$ by using induction on k and the same methods as in the preceding proof.

COROLLARY 1. *If S is a closed subset of E^2 having n' essential lnc points, with $S \sim Q$ connected and $k \geqq 0$ bounded components in the complement of $S \sim Q$, then S is the union of $n' - k + 1$ or fewer closed convex sets.*

Sets in the plane may be easily constructed for which the bounds in the theorem and corollary are realized for each n. See Fig. 3 for an illustration of several special cases; note that the class of examples shown supports the conjecture that for all closed planar sets S with $S \sim Q$ connected, $n' \geqq \frac{1}{2}(n + 1)$.



n = 11, n' = 8 (Union of 9 convex sets)

n = 7, n' = 4 (Union of 5 convex sets)

n = 10, n' = 8, k = 2 (Union of 7 convex sets)

Fig. 3.

4. Bounds for sets having property P_m

In seeking conditions which together with m-convexity imply the existence of a convex covering of S by $m-1$ convex subsets, a likely formula for success would seem to be the study of the nature (cardinality or structure) of the set of lnc points. Indeed, Lemma 6 is highly suggestive; a general property known for some time is that for closed sets with $|Q| \equiv n = 1$, m-convexity implies the existence of a convex covering by $m - 1$ subsets [proved in Guay's doctoral thesis, Michigan State University (1967)]; note that such a covering of S is characteristic of m-convexity since the converse is true. For the case $m = 3$ the same implication holds for closed planar sets if n is even or infinite (Valentine [5]), or if the dimension of S is $\geqq 3$, the interior of the kernel of S is nonempty, and $Q \subset$ int conv S (Buchman [1]). The five-pointed star with interior is a counterexample for the case $m = 3$, $n = 5$. A student of one of the authors, Mr. John Legge, constructed a counterexample for the case $m = 4$, $n = 4$, of which the illustration in Fig. 4 is the authors' version, and the example of the triangle and semicircles on the sides provides a counterexample for the case $m = 3$, $n = 3$. Note that in each of these examples $S \sim Q$ is connected. Thus, the following lemma is a best possible result.

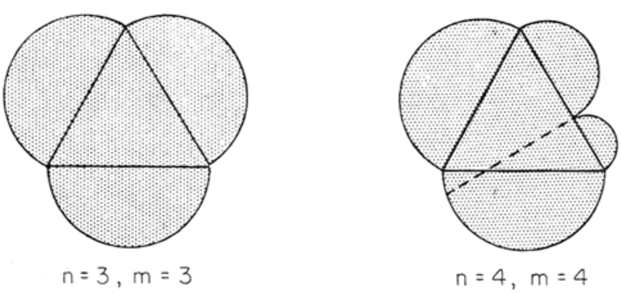


Fig. 4.

LEMMA 9. *If S is closed, $S \sim Q$ is connected, and $n \leqq 2$, then S is m-convex iff it is the union of $m - 1$ convex sets.*

PROOF. The proposition is trivial if either $n = 0$ or $m = 2$; hence we may assume S is planar (Lemma 4). Since Theorem 1 proves the assertion for all $m \geqq 4$ if $n = 2$ and for $m \geqq 3$ if $n = 1$, the only case remaining is $m = 3$, $n = 2$. Theorem 3 of Valentine in [5] now applies and asserts that S is the union of two convex sets.

The next proposition allows us to establish the preceding result for closed sets in general.

LEMMA 10. *Let S be a closed m-convex set with Q finite and nonempty. If no component of $S \sim Q$ is one-dimensional and W is any component different from $S \sim Q$, then clW is exactly k-convex [k-convex but not $(k-1)$-convex] for some $k \leqq m-1$, and $cl(S \sim cl\,W)$ is $(m-k+1)$-convex.*

PROOF. Let $\{W_\alpha : \alpha \in A\}$ denote the family of components of $S \sim Q$, with $W = W_{\alpha_0}$ the given component. Observe that $S = \cup_{\alpha \in A} \operatorname{cl} W_\alpha$ and $\operatorname{cl}(S \sim \operatorname{cl} W) = \cup_{\alpha \neq \alpha_0} \operatorname{cl} W_\alpha$. Suppose there exist points $x_1, \cdots, x_{k-1}$ in $\operatorname{cl} W$ such that $x_i x_j \not\subset \operatorname{cl} W$ for $1 \leqq i < j \leqq k-1$, $k \geqq 2$, and that $x_k, \cdots, x_{k+r-2}$ are points in $\operatorname{cl} S \sim \operatorname{cl} W$ such that $x_i x_j \not\subset \operatorname{cl}(S \sim \operatorname{cl} W)$ for $k \leqq i < j \leqq k+r-2$, $r \geqq 2$. Let W_i denote a component of $S \sim Q$ whose closure contains x_i for $i = 1, \cdots, k+r-2$, with $W_1 = \cdots = W_{k-1} = W$ and $W_i \neq W$ for $i \geqq k$. Since $\operatorname{cl} W$ and $\operatorname{cl}(S \sim \operatorname{cl} W)$ are closed there exist neighborhoods U_i of x_i for each i, with $U_i \cap W = \emptyset$ for $i \geqq k$, such that for any $u_i \in U_i$, $u_i u_j \not\subset \operatorname{cl} W$ for $1 \leqq i < j \leqq k-1$ and $u_i u_j \not\subset \operatorname{cl}(S \sim \operatorname{cl} W)$ for $k \leqq i < j \leqq k+r-2$, provided both $k > 2$ and $r > 2$; if $k = 2$ or $r = 2$ (when either $\operatorname{cl} W$ or $\operatorname{cl}(S \sim \operatorname{cl} W)$ is convex) simply choose U_1 or U_k as any neighborhood containing x_1 or x_k, respectively. Now $\operatorname{cl} W_i$ is either planar or convex. If $\operatorname{cl} W_i$ is planar, Lemma 5 implies that $\operatorname{cl} W_i = \operatorname{cl}(\operatorname{int} W_i)$ (relative to the plane of W_i), so there exists a point $y_i \in U_i \cap \operatorname{int} W_i$ and thus, a circular disk V_i about y_i as center exists such that $V_i \subset W_i \cap U_i$. If $\operatorname{cl} W_i$ is convex let π_i be any plane passing through x_i. Then $\pi_i \cap \operatorname{cl} W_i$ is a two-dimensional convex set containing x_i and there exists $y_i \in U_i \cap \operatorname{int}(\pi_i \cap \operatorname{cl} W_i)$ and a circular disk V_i about y_i as center such that $V_i \subset W_i \cap U_i$ (since y_i may be chosen outside Q and $V_i \cap Q = \emptyset$ may then be assumed). Now choose points $p_i \in \operatorname{int} V_i$ for $1 \leqq i \leqq k+r-2$ as follows: $p_1 \in \operatorname{int} V_1$ is chosen arbitrarily. After having chosen points $p_j \in \operatorname{int} V_j$ for $1 \leqq j < i$, choose $p_i \in \operatorname{int} V_i \sim \cup_{q \in Q} \cup_{j<i} L(q, p_j)$. The points thus chosen clearly satisfy $p_i p_j \cap Q = \emptyset$ for $1 \leqq i < j \leqq k+r-2$.

Thus, if for some $i < j$ $p_i p_j \subset S$, then $p_i p_j \subset S \sim Q$, and therefore $p_i \in W_i$ implies $p_i p_j \subset W_i \subset \operatorname{cl} W_i$ so that $j > i \geqq k$ and $p_i p_j \not\subset \operatorname{cl}(S \sim \operatorname{cl} W)$. Hence, at least one point $w \in W$ lies on $p_i p_j$ and hence $p_i w \subset W$ or $p_i \in W$, a contradiction. It follows that for $i \neq j$, $p_i p_j \not\subset S$. By the m-convexity of S,

$$k + r - 2 \leqq m - 1,$$

or $k \leqq m - r + 1$. Since $r \geqq 2$, $\operatorname{cl} W$ is exactly k-convex with $k \leqq m-1$, and $\operatorname{cl}(S \sim \operatorname{cl} W)$ is r-convex with $r \leqq m - k + 1$.

THEOREM 2. *A closed set S in a topological linear space having at most two* lnc *points is m-convex iff it is the union of $m-1$ convex sets.*

PROOF. The proof will be by induction on m. If $S \sim Q$ has a one-dimensional component W, which must itself be convex, denote by L the line that contains W, and let $\tilde{W}$ be the component of $S \cap L$ that contains W. Then $\tilde{W}$ is convex, and it is clear that $\mathrm{cl}(S \sim \tilde{W})$ is $(m-1)$-convex and hence is the union of $m-2$ convex sets, so that $S = \tilde{W} \cup \mathrm{cl}(S \sim \tilde{W})$ is the union of $m-1$ convex sets. Thus, assume that $S \sim Q$ has no one-dimensional components, and let W be any component of $S \sim Q$ different from $S \sim Q$ (if $S \sim Q$ is connected Lemma 9 makes the desired assertion). By Lemma 10, $\mathrm{cl}W$ is k-convex for some $2 \leqq k \leqq m-1$ and $\mathrm{cl}(S \sim \mathrm{cl}\, W)$ is $(m-k+1)$-convex. Hence, by the induction hypothesis S is the union of

$$(k-1)+(m-k) = m-1$$

or fewer convex sets.

A simple induction argument may be used to establish the following generalization of Theorem 2:

THEOREM 3. *If S is a closed m-convex set in a topological linear space and S has n points of local nonconvexity, where $0 < n < \infty$, then S is the union of k convex sets, where*

$$k \leqq (m-1)\left[\frac{n+1}{2}\right].$$

PROOF. For $n \leqq 2$, Theorem 3 reduces to Theorem 2. Assuming $n \geqq 3$, let $W_1, \cdots, W_r$ be the components of $S \sim Q$ (Lemma 10 implies $r < \infty$). By Lemma 10 $\mathrm{cl}\, W_i$ is m_i-convex where $2 \leqq m_i \leqq m-1$ and $\sum_{i=1}^{i=r}(m_i - 1) = m-1$. If Q_i denotes the set of lnc points of $\mathrm{cl}\, W_i = S_i$, it follows that $|Q_i| \equiv n_i \leqq n$ and $S_i \sim Q_i$ is connected. If $r = 1$ the set S is planar and for $m = 3$ Valentine's theorem [5] gives us

$$k \leqq 3 \leqq n \leqq 2\left[\frac{n+1}{2}\right] = (m-1)\left[\frac{n+1}{2}\right],$$

while for $m \geqq 4$ Theorem 1 implies

$$k \leqq n+1 < 3\left[\frac{n+1}{2}\right] \leqq (m-1)\left[\frac{n+1}{2}\right].$$

Finally, if $r > 1$ the induction hypothesis implies that S is the union of

$$k \leqq \sum_{i=1}^{i=r} (m_i - 1)\left[\frac{n_i+1}{2}\right] \leqq \sum_{i=1}^{i=r} (m_i - 1)\left[\frac{n+1}{2}\right] = (m-1)\left[\frac{n+1}{2}\right]$$

convex sets.

The referee has pointed out that for planar sets a much better bound may be obtained:

THEOREM 4. *If S is a closed m-convex subset of E^2 having n essential* lnc *points, then S is the union of k convex sets, where*

$$k \leqq m + n - 1.$$

PROOF. As in the previous theorem let $W_1, \cdots, W_r$ be the components of $S \sim Q$. If $r = 1$, Theorem 1 implies that S is the union of

$$k \leqq n + 1 \leqq m + n - 1$$

convex sets. For $r > 1$, the induction hypothesis implies that $S_i = \mathrm{cl}\, W_i$ is the union of $m_i + n_i - 1$ convex sets. Since no point $q \in Q$ can be an essential lnc point for more than one set S_i we have $\sum_{i=1}^{i=r} n_i = n$ and therefore

$$k \leqq \sum_{i=1}^{i=r} (m_i + n_i - 1) = m + n - 1.$$

REFERENCES

1. E. Buchman, *Generalizations of arcwise convex sets*, Doctoral Dissertation, University of California, Los Angeles, 1968.

2. D. C. Kay and M. D. Guay, *Convexity and a certain property P_m*, Israel J. Math. **8** (1970), 39–52.

3. J. L. Kelley and I. Namioka, *Linear topological spaces*, Van Nostrand, Princeton, New Jersey, 1963.

4. C. Kuratowski, *Topologie I*, 2nd ed., Monogr. Mat., vol. 20, P. W. N., Warsaw, 1948.

5. F. A. Valentine, *A three point convexity property*, Pacif. J. Math. **7** (1957), 1227–1235.

6. F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

7. F. A. Valentine, *Local convexity and L_n sets*, Proc. Amer. Math. Soc. **16** (1965), 1305–1310.

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